

ALMOST NEWTON, SOMETIMES LATTÈS

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ABSTRACT. Self-maps everywhere defined on the projective space \mathbb{P}^N over a number field or a function field are the basic objects of study in the arithmetic of dynamical systems. One reason is a theorem of Fakhruddin [4] (with complements in [1]) that asserts that a “polarized” self-map of a projective variety is essentially the restriction of a self-map of the projective space given by the polarization. In this paper we study the natural self-maps defined the following way: F is a homogeneous polynomial of degree d in $(N + 1)$ variables X_i defining a smooth hypersurface. Suppose the characteristic of the field does not divide d and define the map of partial derivatives $\phi_F = (F_{X_0}, \dots, F_{X_N})$. The map ϕ_F is defined everywhere due to the following formula of Euler: $\sum X_i F_{X_i} = dF$, which implies that a point where all the partial derivatives vanish is a non-smooth point of the hypersurface $F = 0$. One can also compose such a map with an element of PGL_{N+1} . In the particular case addressed in this article, $N = 1$, the smoothness condition means that F has only simple zeroes. In this manner, fixed points and their multipliers are easy to describe and, moreover, with a few modifications we recover classical dynamical systems like the Newton method for finding roots of polynomials or the Lattès map corresponding to the multiplication by 2 on an elliptic curve.

1. INTRODUCTION

Self-maps everywhere defined on the projective space \mathbb{P}^N over a number field or a function field are the basic objects of study in the arithmetic of dynamical systems. One reason is a theorem of Fakhruddin [4] (with complements in [1]) that asserts that a “polarized” self-map of a projective variety is essentially the restriction of a self-map of the projective space given by the polarization. In this paper we study the natural self-maps defined the following way: F is a homogeneous polynomial of degree d in $(N + 1)$ variables X_i defining a smooth hypersurface. Suppose the characteristic of the field does not divide d and define the map of partial derivatives $\phi_F = (F_{X_0}, \dots, F_{X_N})$. The map ϕ_F is defined everywhere due to the following formula of Euler:

$$\sum X_i F_{X_i} = dF,$$

which implies that a point where all the partial derivatives vanish is a non-smooth point of the hypersurface $F = 0$. One can also compose such a map with an element of PGL_{N+1} . In the particular case addressed in this article, $N = 1$, the smoothness condition means that F has only simple zeroes. In this manner, fixed points and their multipliers are easy to describe and, moreover, with a few modifications we recover classical dynamical systems like the Newton method for finding roots of polynomials or the Lattès map corresponding to the multiplication by 2 on an elliptic curve. We begin by recalling some of the definitions and objects we need from dynamical systems before stating the main results.

Given a morphism $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ we can iterate ϕ to create a (discrete) dynamical system. We denote the n^{th} iterate of ϕ as $\phi^n = \phi(\phi^{n-1})$. Calculus students are exposed to dynamical systems

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through the iterated root finding method known as Newton's Method where, given a differentiable function $f(x)$ and an initial point x_0 , one constructs the sequence

$$x_{n+1} = \phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In general, this sequence converges to a root of $f(x)$. In terms of dynamics, we would say that the roots of $f(x)$ are attracting fixed points of $\phi(x)$. More generally, one says that P is a *periodic point of period n* for ϕ if $\phi^n(P) = P$.

A common example of a dynamical system with periodic points is to take an endomorphism of an elliptic curve $[m] : E \rightarrow E$ and project onto the first coordinate. This construction induces a map on \mathbb{P}^1 called a *Lattès map*, and for $m \in \mathbb{Z}$ its degree is m^2 and its periodic points are the torsion points of the elliptic curve.

Denote Hom_d as the set of degree d morphisms on \mathbb{P}^1 . There is a natural action on \mathbb{P}^1 by PGL_2 through conjugation that induces an action on Hom_d . We take the quotient as $M_d = \text{Hom}_d / \text{PGL}_2$. By [9], the moduli space M_d is a geometric quotient. We say that $\gamma \in \text{PGL}_2$ is an *automorphism* of $\phi \in \text{Hom}_d$ if $\gamma^{-1} \circ \phi \circ \gamma = \phi$. We denote the (finite [7]) group of automorphisms as $\text{Aut}(\phi)$.

Let K be a number field and $F \in K[X, Y]$ be a homogeneous polynomial of degree d with distinct roots. Define

$$\phi_F(X, Y) = [F_Y, -F_X] : \mathbb{P}^1 \rightarrow \mathbb{P}^1.$$

Remark. We can think of ϕ_F as the linear combination

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} F_X \\ F_Y \end{pmatrix}.$$

It may also be interesting to study other families of linear combinations of the partial derivatives arising from other elements of GL_2 , but we do not address them in this article.

In Section 2 we examine the dynamical properties of these maps.

Theorem (Theorem 5). *The fixed points of $\phi_F(X, Y)$ are the solutions to $F(X, Y) = 0$, and the multipliers of the fixed points are $1 - d$.*

Theorem (Theorem 6 and Corollary 7). *The family of maps of the form $\phi_F = (F_Y, -F_X) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is invariant under the conjugation action by PGL_2 .*

We also give a description of the higher order periodic points and a recursive definition of the polynomial whose roots are the n -periodic points. We examine related, more general Newton-Raphson maps and, finally, recall the connection to invariant theory and maps with automorphisms.

In Section 3 we explore the connection with Lattès maps.

Theorem (Theorem 17). *Maps of the form*

$$\tilde{\phi}(x) = x - 3 \frac{f(x)}{f'(x)}$$

are the Lattès maps from multiplication by [2] and $f(x) = \prod (x - x_i)$ where x_i are the x -coordinates of the 3-torsion points.

Finally, when E has complex multiplication ($m \notin \mathbb{Z}$) the associated ϕ_F can have a non-trivial automorphism group.

Theorem (Theorem 18). *If E has $\text{Aut}(E) \supsetneq \mathbb{Z}/2\mathbb{Z}$ and the zeros of $F(X, Y)$ are torsion points of E , then an induced map ϕ_F has a non-trivial automorphism group.*

2. ALMOST NEWTON MAPS

Let K be a field and consider a two variable homogeneous polynomial $F(X, Y) \in K[X, Y]$ of degree d with no multiple roots. Consider the degree $d - 1$ map

$$\begin{aligned}\phi_F : \mathbb{P}^1 &\rightarrow \mathbb{P}^1 \\ (X, Y) &\mapsto (F_Y(X, Y), -F_X(X, Y)).\end{aligned}$$

In particular, $F_X = F_Y = 0$ has no nonzero solutions and so ϕ_F is a morphism. Label $x = \frac{X}{Y}$ and consider

$$f(x) = \frac{F(X, Y)}{Y^d}$$

and notice that

$$f'(x) = \frac{F_X(X, Y)}{Y^{d-1}}.$$

Lemma 1. *The map induced on affine space by ϕ_F is given by*

$$\tilde{\phi}_F(x) = x - d \frac{f(x)}{f'(x)}.$$

Proof.

$$\tilde{\phi}_F(x) = -\frac{F_Y(X, Y)}{F_X(X, Y)} = -\frac{Y F_Y(X, Y)}{Y F_X(X, Y)} = \frac{X F_X(X, Y) - d F(X, Y)}{Y F_X(X, Y)} = x - d \frac{f(x)}{f'(x)}.$$

□

Definition 2. Let $\phi = (\phi_1, \phi_2) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a rational map on \mathbb{P}^1 . Define $\text{Res}(\phi) = \text{Res}(\phi_1, \phi_2)$, the *resultant* of the coordinate functions of ϕ . For a homogeneous polynomial F , denote $\text{Disc}(F)$ for the *discriminant* of F .

Proposition 3. *Let $F(X, Y)$ be a homogeneous polynomial of degree d with no multiple roots. Then,*

$$\text{Res}(\phi_F(X, Y)) = (-1)^{d(d-1)/2} d^{d-2} \text{Disc}(F(X, Y)).$$

Proof. Denote $F(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + \dots + a_0 Y^d$. Then we have

$$\begin{aligned}F_X(X, Y) &= d a_d X^{d-1} + \dots + a_1 Y^{d-1} \\ F_Y(X, Y) &= a_{d-1} X^{d-1} + \dots + d a_0 Y^{d-1}.\end{aligned}$$

From standard properties of resultants and discriminants we have

$$\begin{aligned}a_d \text{Disc}(F(X, Y)) &= (-1)^{d(d-1)/2} \text{Res}(F(X, Y), F_X(X, Y)) \\ &= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} \text{Res}(dF(X, Y), -F_X(X, Y)) \\ &= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} \text{Res}(X F_X(X, Y) + Y F_Y(X, Y), -F_X(X, Y)) \\ &= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} \text{Res}(Y F_Y(X, Y), -F_X(X, Y)).\end{aligned}$$

Now we see that

$$\text{Res}(YF_Y, -F_X) = \begin{vmatrix} 0 & a_{d-1} & 2a_{d-2} & \cdots & da_1 & 0 \\ 0 & 0 & a_{d-1} & 2a_{d-2} & \cdots & da_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -da_d & -(d-1)a_{d-1} & \cdots & -a_1 & 0 & 0 \\ 0 & -da_d & -(d-1)a_{d-1} & \cdots & -a_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}.$$

Expanding down the first column we have

$$\begin{aligned} & \text{Res}(YF_Y(X, Y), -F_X(X, Y)) \\ &= -da_n(-1)^{d+1} \begin{vmatrix} a_{d-1} & 2a_{d-2} & \cdots & da_1 & 0 & 0 \\ 0 & a_{d-1} & 2a_{d-2} & \cdots & da_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -da_d & -(d-1)a_{d-1} & \cdots & -a_1 & 0 & 0 \\ 0 & -da_d & -(d-1)a_{d-1} & \cdots & -a_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} \\ &= da_d(-1)^{d+2} R(F_Y(X, Y), -F_X(X, Y)). \end{aligned}$$

Thus, we compute

$$\begin{aligned} a_d \text{Disc}(F(X, Y)) &= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} \text{Res}(YF_Y(X, Y), -F_X(X, Y)) \\ &= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} (-1)^{d+2} da_n \text{Res}(F_Y(X, Y), -F_X(X, Y)) \\ &= (-1)^{d(d-1)/2} \frac{a_d}{d^{d-2}} \text{Res}(F_Y(X, Y), -F_X(X, Y)). \end{aligned}$$

□

Remark. The similar relationship for flexible Lattès maps ($[m]$ for $m \in \mathbb{Z}$)

$$\text{Disc}(\Psi_{E,m-1} \Psi_{E,m+1}) = c \text{Res}(\phi_{E,m}),$$

where $\Psi_{E,m}$ is the m -division polynomial and $\phi_{E,m}$ is the Lattès map induced by $[m]$, seems to not be currently known. Using conjectures on $\text{Disc}(\Psi_{E,m})$ from [2] and the formula for $\text{Res}(\phi_{E,m})$ [10, Exercise 6.23] it appears that the exponent is correct and that constant should be

$$c = \pm 2^a (m-1)^b (m+1)^c$$

for some integers a, b, c . It would be interesting to determine the exact relation.

Definition 4. Let P be a periodic point of period n for $\tilde{\phi}$, then the *multiplier* at P is the value $(\tilde{\phi}^n)'(P)$. If P is the point at infinity, then we can compute the multiplier by first changing coordinates.

Theorem 5. The fixed points of $\phi_F(X, Y)$ are the solutions to $F(X, Y) = 0$, and the multipliers of the fixed points are $1 - d$.

Proof. The projective equality

$$\phi(X, Y) = (X, Y)$$

is equivalent to

$$YF_Y(X, Y) = -XF_X(X, Y).$$

Using the formula of Euler for homogeneous polynomials we then have

$$XF_X(X, Y) + YF_Y(X, Y) = dF(X, Y) = 0.$$

Since d is a nonzero integer the fixed points satisfy $F(X, Y) = 0$.

To calculate the multipliers, we first examine the affine fixed points. We take a derivative evaluated at a fixed point to see

$$\tilde{\phi}'_F(x) = 1 - d \frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = 1 - d \frac{f'(x)f'(x)}{(f'(x))^2} = 1 - d.$$

If a fixed point has multiplier one, then it would have multiplicity at least 2 and, hence, would be at least a double root of F . Since F has no multiple roots, every multiplier is not equal to one. Thus, to see that the multiplier at infinity (when it is fixed) is also $1 - d$ we may use the relation [10, Theorem 1.14]

$$(1) \quad \sum_{i=1}^d \frac{1}{1 - \lambda_i} = 1.$$

□

Remark. If $\text{char } K \mid d$, then ϕ_F is the identity map. Let $F(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + \cdots + a_0 Y^d$. Then we have

$$\begin{aligned} F_X(X, Y) &= (d-1)a_{d-1}X^{d-1}Y + \cdots + a_1Y^{d-1} = Y((d-1)a_{d-1}X^{d-1} + \cdots + a_1Y^{d-2}) \\ F_Y(X, Y) &= a_{d-1}X^{d-1} + \cdots + (d-1)a_1Y^{d-2}X = X(a_{d-1}X^{d-1} + \cdots + (d-1)a_1Y^{d-2}). \end{aligned}$$

Since $-i \equiv d - i \pmod{d}$ for $0 \leq i \leq d$ we have that

$$\phi_F(X, Y) = (F_Y, -F_X) = (XP(X, Y), YP(X, Y)) = (X, Y),$$

where $P(X, Y)$ is a homogeneous polynomial.

We next show that maps of the form ϕ_F form a family in the moduli space of dynamical systems. In other words, for every $\gamma \in \text{PGL}_2$ and ϕ_F , there exists a $G(X, Y)$ such that $\gamma^{-1} \circ \phi_F \circ \gamma = \phi_G$. In fact, $G(X, Y)$ is the polynomial that results from allowing γ^{-1} to act on F .

Theorem 6. *Every rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $d-1$ whose fixed points are $\{(a_1, b_1), \dots, (a_d, b_d)\}$ all with multiplier $(1 - d)$ is a map of the form $\phi_F(X, Y) = (F_Y(X, Y), -F_X(X, Y))$ for*

$$F(X, Y) = (b_1X - a_1Y)(b_2X - a_2Y) \cdots (b_dX - a_dY).$$

Proof. Let $(a_1, b_1), \dots, (a_d, b_d)$ be the collection of fixed points for the map $\psi(X, Y) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ whose multipliers are $1 - d$. Then on \mathbb{A}^1 we may write the map of degree $d - 1$ as

$$\tilde{\psi}(x) = x - \frac{P(x)}{Q(x)}$$

for some pair of polynomials $P(x)$ and $Q(x)$ with no common zeros. Let $\tilde{\phi}_F(x)$ be the affine map associated to $F(X, Y) = (b_1X - a_1Y) \cdots (b_dX - a_dY)$ and we can write

$$\tilde{\phi}_F(x) = x - d \frac{f(x)}{f'(x)}$$

where

$$f(x) = \frac{F(X, Y)}{Y^d}.$$

The fixed points of $\tilde{\psi}(x)$ are the points where $\frac{P(x)}{Q(x)} = 0$ and, hence, where $P(x) = 0$. The fixed points of $\tilde{\psi}(x)$ are the same as for $\tilde{\phi}_F(x)$, so we must have $P(x) = cf(x)$ for some nonzero constant c . Using the fact that the multipliers are $1 - d$, we get

$$\tilde{\psi}'(x) = 1 - \frac{cf'Q - cQ'}{(Q')^2} = 1 - \frac{cf'}{Q} = 1 - d.$$

Therefore, we know that

$$\frac{c}{d}f'(x_i) = Q(x_i)$$

where x_1, \dots, x_d are the fixed points (or x_1, \dots, x_{d-1} if $(1, 0) \in \mathbb{P}^1$ is a fixed point). Since $f'(x)$ and $Q(x)$ are both degree $d - 1$ polynomials (or $d - 2$), this is a system of d (or $d - 1$) equations in the d (or $d - 1$) coefficients of $Q(x)$. Since the values x_i are distinct (since the multipliers are $\neq 1$) the Vandermonde matrix is invertible and we get a unique solution for $Q(x)$. In particular, we must have

$$\frac{c}{d}f'(x) = Q(x)$$

and thus

$$\tilde{\psi}(x) = \tilde{\phi}(x).$$

□

Corollary 7. *The family of maps of the form $\phi_F(X, Y) = (F_Y(X, Y), -F_X(X, Y)) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is invariant under the conjugation action by PGL_2 . In particular, the family of ϕ_F where $\deg F(X, Y) = d$ is isomorphic to an arbitrary choice of $d - 3$ distinct points in $\mathbb{P}^1 - \{0, 1, \infty\}$.*

Proof. Conjugation fixes the multipliers and moves the fixed points, so by Theorem 6 the conjugated map is of the same form.

A map of degree $d - 1$ on \mathbb{P}^1 has d fixed points. The action by PGL_2 can move any 3 distinct points to any 3 distinct points. Thus, the choice of the remaining $d - 3$ fixed points determines ϕ_F . □

2.1. Extended Example.

Proposition 8. *Let $F(X, Y)$ be a degree 4 homogeneous polynomial with no multiple roots with associated morphism $\phi_F(X, Y)$. For any $\alpha \in \overline{\mathbb{Q}} - \{0, 1\}$ we have that $\phi_F(X, Y)$ is conjugate to a map of the form*

$$\phi_{F,\alpha}(X, Y) = (X^3 - 2(\alpha + 1)X^2Y + 3\alpha XY^2, -3X^2Y + 2(\alpha + 1)XY^2 - \alpha Y^3).$$

Proof. We can move three of the 4 fixed points to $\{0, 1, \infty\}$ with an element of PGL_2 and label the fourth fixed point as α . Then we have

$$F(X, Y, \alpha) = (X)(Y)(X - Y)(X - \alpha Y) = X^3Y - (\alpha + 1)X^2Y^2 + \alpha XY^3$$

and

$$\begin{aligned} \phi_{F,\alpha}(X, Y) &= (F_Y(X, Y, \alpha), -F_X(X, Y, \alpha)) \\ &= (X^3 - 2(\alpha + 1)X^2Y + 3\alpha XY^2, -(3X^2Y - 2(\alpha + 1)XY^2 + \alpha Y^3)). \end{aligned}$$

□

Proposition 9. *Let $F(X, Y)$ be a degree 4 homogeneous polynomial with no multiple roots with associated morphism $\phi_F(X, Y)$. Assume that $\phi_F(X, Y)$ is in the form of Proposition 8. Then, the two periodic points are of the form*

$$\{\pm\sqrt{\alpha}, 1 \pm \sqrt{1 - \alpha}, \alpha \pm \sqrt{\alpha^2 - \alpha}\} \cup \{0, 1, \infty, \alpha\}.$$

Proof. Direct computation. □

Proposition 10. \mathbb{Q} -Rational affine two periodic points are parameterized by pythagorean triples.

Proof. The values α and $1 - \alpha$ are both squares and $0 < \alpha < 1$. Thus, there are relatively prime integers p and q so that $\alpha = \frac{p^2}{q^2}$ with $p < q$ and $1 - \alpha = \frac{q^2 - p^2}{q^2}$. Therefore, $r^2 + p^2 = q^2$ is a pythagorean triple, with $r^2 = (1 - \alpha)q^2$. \square

Remark. The 2-periodic points are not the roots of $f(\tilde{\phi}(x))$, see Theorem 12 for the general relation.

For general $F(X, Y)$, $\phi_F^2(X, Y)$ does not come from a homogeneous polynomial G .

2.2. Higher order periodic points. We set the following notation

$$f(x) = \frac{F(X, Y)}{Y^d} = \sum_{i=0}^{d-1} a_i x^i$$

$$\tilde{\phi}^n(x) = \frac{A_n(x)}{B_n(x)}$$

$$c_n = -\frac{B_{n+1}(x)}{F_X(A_n(x), B_n(x))}$$

where $A_n(x)$ and $B_n(x)$ are polynomials and c_n is a constant.

Definition 11. Let $\Psi_n(x)$ be the polynomial whose zeros are affine n -periodic points.

The polynomial $\Psi_n(x)$ is the equivalent of the n -division polynomial for elliptic curves, see [6, Chapter 2] for information on division polynomials.

While it is possible, to define $\Psi_n(x)$ recursively, the relation is not as simple as for elliptic curves. If we let $\Psi_{E,m}$ be the m -division polynomial for an elliptic curve E , then

$$\Psi_{E,2m+1} = \Psi_{E,m+2}\Psi_{E,m}^3 - \Psi_{E,m-1}\Psi_{E,m+1}^3 \quad \text{for } m \geq 2$$

$$\Psi_{E,2m} = \left(\frac{\Psi_{E,m}}{2y}\right) (\Psi_{E,m+2}\Psi_{E,m-1}^2 - \Psi_{E,m-2}\Psi_{E,m+1}^2) \quad \text{for } m \geq 3.$$

Notice that these relations depend only on $\Psi_{E,m}$ for various m , whereas the formula in the following theorem also involves iterates of the map.

Theorem 12. *We have the following formulas*

$$\tilde{\phi}^n(x) = x + d \frac{\Psi_n(x)}{B_n(x)}$$

and

$$\Psi_{n+1}(x) = \frac{F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))}{B_n(x)c_n}$$

with multipliers

$$\prod_{i=0}^{n-1} \left(1 - d + d \frac{f(\phi^i(x))f''(\phi^i(x))}{f'(\phi^i(x))^2}\right).$$

Proof. We proceed inductively. For $n = 1$ we know that the fixed points are the zeros of $f(x)$.

$$\tilde{\phi}(x) = x - d \frac{f(x)}{f'(x)} = x - d \frac{f(x)}{F_X(A_0(x), B_0(x))} = x - d \frac{f(x)}{-B_1(x)} = x + d \frac{\Psi_1(x)}{B_1(x)}.$$

Now assume that

$$\tilde{\phi}^n(x) = x + d \frac{\Psi_n(x)}{B_n(x)}.$$

Computing

$$\begin{aligned}
\tilde{\phi}^{n+1}(x) &= x + d \frac{\Psi_n(x)}{B_n(x)} - d \frac{f(\tilde{\phi}^n(x))}{f'(\tilde{\phi}^n(x))} \\
&= x + d \frac{\Psi_n(x)}{B_n(x)} - d \frac{F(A_n(x), B_n(x))}{B_n(x)F_X(A_n(x), B_n(x))} \\
&= x - d \frac{F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))}{B_n(x)F_X(A_n(x), B_n(x))} \\
&= x - d \frac{F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))}{c_n B_n(x)B_{n+1}(x)}.
\end{aligned}$$

So we have to show that $B_n(x)$ divides $F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))$. Working modulo $B_n(x)$ we see that

$$F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x)) \equiv A_n(x)^d - (A_n(x)/d)dA_n(x)^{d-1} \equiv 0 \pmod{B_n(x)}$$

where we used the induction assumption for $\Psi_n(x)$. Thus, the n -periodic points are among the roots of $\Psi_n(x)$.

For equivalence, we count degrees. Again, proceeding inductively it is clear for $n = 1$. For $n + 1$ we have that

$$\deg(F(A_n(x), B_n(x))) = d(d-1)^n = (d-1)^{n+1} + (d-1)^n$$

and

$$\deg(\Psi_n(x)F_X(A_n(x), B_n(x))) \leq (d-1)^n + 1 + (d-1)^{n+1}$$

depending on whether the point at infinity is periodic or not. Thus,

$$\deg(\Psi_{n+1}(x)) \leq (d-1)^n + 1 + (d-1)^{n+1} - (d-1)^n = (d-1)^{n+1} + 1.$$

Since the number of (projective) periodic points of ϕ^n is $(d-1)^n + 1$, every affine fixed point must be a zero of $\Psi_n(x)$.

We compute the multipliers as

$$\begin{aligned}
\tilde{\phi}'(x) &= 1 - d \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = 1 - d + d \frac{f(x)f''(x)}{f'(x)^2} \\
(\tilde{\phi}^n(x))' &= \prod_{i=0}^{n-1} \tilde{\phi}'(\tilde{\phi}^i(x)) = \prod_{i=0}^{n-1} \left(1 - d + d \frac{f(\tilde{\phi}^i(x))f''(\tilde{\phi}^i(x))}{f'(\tilde{\phi}^i(x))^2} \right).
\end{aligned}$$

□

2.3. Replace d with r : Modified Newton-Raphson Iteration. We have considered maps of the form

$$\tilde{\phi}_F(x) = x - d \frac{f(x)}{f'(x)}$$

where $d = \deg(F(X, Y))$. However, we could also consider affine maps of the form

$$(2) \quad \tilde{\phi}(x) = x - r \frac{f(x)}{f'(x)}$$

for some $r \neq 0$ and polynomial $f(x)$. When used for iterated root finding, such maps are often called the modified Newton-Raphson method. The fixed points are again the zeros of $f(x)$ and are

all distinct with multipliers $1 - r$. Thus, if $\deg f \neq r$, then the point at infinity must also be a fixed point by (1) with multiplier

$$\sum_{i=1}^{d+1} \frac{1}{1 - \lambda_i} = \frac{\deg f(x)}{r} + \frac{1}{1 - \lambda_\infty} = 1$$

$$\lambda_\infty = \frac{\deg f(x)}{\deg f(x) - r}.$$

These maps also form a family in the moduli space of dynamical systems and are determined by their fixed points..

Theorem 13. *Let r be a non-zero integer. Every rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $d - 1$ which has $d - 1$ affine fixed points all with multiplier $(1 - r)$ and fixes $(1, 0)$ with multiplier $\frac{d-1}{d-r-1}$ is a map of the form (2).*

Proof. The method of proof is identical to the proof of Theorem 6, so is omitted. \square

Remark. Note that if we choose $r = 1$, then all of the affine fixed points are also critical points ($\tilde{\phi}'(x) = 0$) as noted in [3, Corollary 1].

2.4. Connection to Maps with Automorphisms. Let $\Gamma \subset \text{PGL}_2$ be a finite group.

Definition 14. We say that a homogeneous polynomial F is an *invariant* of Γ if $F \circ \gamma = \chi(\gamma)F$ for all $\gamma \in \Gamma$ and some character χ of Γ . The *invariant ring* of Γ denoted $K[X, Y]^\Gamma$ is the set of all invariants.

The following was known as early as [5, footnote p.345].

Theorem 15. *If $F(X, Y)$ is a homogeneous invariant of a finite group $\Gamma \subset \text{PGL}_2$, then $\Gamma \subset \text{Aut}(\phi_F)$.*

Proof. Easy application of the chain rule. \square

3. CONNECTION TO LATTÈS MAPS

Consider an elliptic curve with Weierstrass equation $E : y^2 = g(x)$ for $g(x) = x^3 + ax^2 + bx + c$. The solutions $g(x) = 0$ are the 2-torsion points. If we integrate $g(x)$ we get $G(x) = x^4/4 + a/3x^3 + b/2x^2 + cx + C$ for some constant C . If we let $C = -(b^2 - 4ac)/12$, then the solutions $G(x) = 0$ are the 3-torsion points.

Recall that a Lattès map is the induced rational function on the first coordinate of the multiplication map $[m] \in \text{End}(E)$ on the rational points of an elliptic curve E ; $\phi_{E,m}(x(P)) = x([m]P)$. For integers $m \geq 3$ we have

$$[m](x, y) = \left(x - \frac{\Psi_{E,m-1}\Psi_{E,m+1}}{\Psi_{E,m}^2}, \frac{\Psi_{E,m+2}\Psi_{E,m-1}^2 - \Psi_{E,m-2}\Psi_{E,m+1}^2}{4y\Psi_{E,m}^3} \right).$$

In other words, the induced Lattès map is given by

$$\phi_{E,m}(x) = x - \frac{\Psi_{E,m-1}\Psi_{E,m+1}}{\psi_m^2}.$$

Hence the fixed points of the Lattès maps are the x -coordinates of the $m - 1$ and $m + 1$ torsion points. For $m = 2$, the fixed points of $\phi_{E,2}$ are the 3 torsion points.

Example 16. Given an elliptic curve of the form $y^2 = g(x) = x^3 + ax^2 + bx + c$. The 2-torsion points satisfy $y^2 = 0$, so are fixed points of the map derived from homogenizing $g(x)$.

$$\begin{aligned} F(X, Y) &= X^3 + aX^2Y + bXY^2 + XY^3 \\ \phi_F(X, Y) &= (aX^2 + 2bXY + 3XY^2, -(2aX + bY^2)) \end{aligned}$$

The fixed points of the doubling map are the points where $x([2]P) = x(P)$, in other words, the points of order 3. They are the points which satisfy the equation

$$\Psi_{E,3}(x) = 3x^4 + 4ax^3 + 6bx^2 + 12cx + (4ac - b^2) = 2g(x)g''(x) - (g'(x))^2$$

So we have

$$\begin{aligned} F(X, Y) &= 3X^4 + 4aX^3Y + 6bX^2Y^2 + 12cXY^3 + (4ac - b^2)Y^4 \\ \phi_F(X, Y) &= (4aX^3 + 12bX^2Y + 36cXY^2 + 4(4ac - b^2)Y^3, \\ &\quad - (12X^3 + 12aX^2Y + 12bXY^2 + 12cY^3)). \end{aligned}$$

For $m = 2$ we get the following stronger result, connecting generalized ϕ_F and Lattès maps.

Theorem 17. *Maps of the form*

$$\tilde{\phi}(x) = x - 3 \frac{f(x)}{f'(x)}$$

are the Lattès maps from multiplication by $[2]$ and $f(x) = \prod (x - x_i)$ where x_i are the x -coordinates of the 3-torsion points.

Proof. From [10, Proposition 6.52] we have the multipliers are all -2 except at ∞ where it is 4 and the fixed points are the 3 torsion points (plus ∞). Now apply Theorem 13. \square

3.1. Complex Multiplication and Automorphisms. For an elliptic curve E , every automorphism is of the form $(x, y) \mapsto (u^2x, u^3y)$ for some $u \in \mathbb{C}^*$ [8, III.10]. In general, the only possibilities are $u = \pm 1$ and $\text{Aut}(E) \cong \mathbb{Z}/2\mathbb{Z}$. However, in the case of complex multiplication $\text{End}(E) \supsetneq \mathbb{Z}$ and it is possible to contain additional roots of unity, thus having $\text{Aut}(E) \supsetneq \mathbb{Z}/2\mathbb{Z}$. The two cases are $j(E) = 0, 1728$ having $\text{Aut}(E) \cong \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$ respectively [8, III.10]. These additional automorphisms induce a linear action $x \mapsto u^2x$ which fixes a polynomial whose roots are torsion points. Thus, the corresponding map ϕ_F has a non-trivial automorphism of the form

$$\begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2.$$

Thus we have shown the following theorem.

Theorem 18. *If E has $\text{Aut}(E) \supsetneq \mathbb{Z}/2\mathbb{Z}$ and the zeros of $F(X, Y)$ are torsion points of E , then an induced map ϕ_F has a non-trivial automorphism group.*

Example 19. Let $E = y^2 = x^3 + ax$, for $a \in \mathbb{Z}$, then $j(E) = 1728$ and $\text{End}(E)$ contains the map $(x, y) \mapsto (-x, iy)$. Thus, the automorphism group of every ϕ_F coming from torsion points satisfies

$$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subset \text{Aut}(\phi_F).$$

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